Riemannian submersions endowed with a semi-symmetric metric connection

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Abstract

In this paper we investigate Riemannian submersions from a Riemannian manifold with a semi-symmetric metric connection onto a Riemannian manifold. We obtain O'Neill's tensor fields for semi-symmetric metric connection. We show that this tensors are not skew symmetric. We obtain derivatives of those tensor fields and compare curvatures of the total manifold, the base manifold and the fibres by computing curvatures.

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1 Introduction

Friedman and Schouten were the first introducing the semi-symmetric linear connection on a manifold M approximately a century ago [3]. Later, Hayden studied on semi-symmetric metric connection [5]. Afterward, Yano investigated a Riemannian manifold with a new connection called a semi-symmetric metric connection [10].

A conventional way to compare two manifolds is by defining smooth maps from one manifold to another. One such map is submersion, whose rank equals to the dimension of the target manifold. An isometric submersion is called a Riemannian submersion. Riemannian submersion between Riemannian manifolds was first studied by O' Neill and Gray [6, 4]. These studies were extended to manifolds with differentiable structures. The theory of Riemannian submersions can be used actively in several areas regarding many applications [8].

In 2018, different connections in Riemannian submersions were investigated for the first time by Akyol and Beğendi [1]. In their study, the basic properties and curvature relations of O'Neill tensors were investigated using semi-symmetric non-metric connection on the basis of basic concepts for Riemannian submersions. Recently, Demir and Sarı defined and studied Riemannian submersions with quarter symmetric metric connection and investigated geometry of new submersions [2]. Moreover, Sarı studied semi-invariant Riemannian submersions with semi-symmetric non-metric connection [9].

In this paper, we investigate for the first time a Riemannian submersion from a Riemannian manifold on the basis of a semi-symmetric metric connection onto a Riemannian manifold with Levi-Civita connection. In Section 2, we review some the standard facts on Riemannian submersions. In section 3, we obtaine basic formulas considering Reimannian submersions and compared with Riemannian manifolds with Levi-Civita connections. Then we could reach the counterparts of O'Neill's tensor fields in the semi-symmetric metric connections. Therefore, we could find basic derivation properties for Reimann submersions. Finally, in section 4, we also investigate curvature

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relations between a Riemannian manifold with semi-symmetric metric connection and a Riemannian manifold with a Levi-Civita connection.

2 Preliminaries

Let M be an n-dimensional Riemannian manifold with Riemannian metric g. We define a linear connection $\tilde{\nabla}$ on a Riemannian manifold M by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)P \tag{2.1}$$

where P is any vector field and η is a 1- form associated with the vector field U on M by

$$\eta(Y) = g(U,Y).$$

Using (2.1), the torsion tensor $\tilde{T}$ of M with respect the connection $\tilde{\nabla}$ is given by

$$\tilde{T}(X,Y) = (\nabla_X Y + \eta(Y)X - g(X,Y)P) - (\nabla_Y X + \eta(X)Y - g(Y,X)P) - [X,Y].$$

Then we have

$$T_X Y = \eta(Y)X - \eta(X)Y \tag{2.2}$$

A linear connection satisfying (2.2) is called a semi-symmetric connection. Moreover, by using (2.1), we obtain

$$(\tilde{\nabla}_X g)(Y,Z) = \tilde{\nabla}_X g(Y,Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) \tag{2.3}$$

for vector fields X, Y, Z on M .

A linear connection $\tilde{\nabla}$ defined by (2.1) satisfies (2.2) and (2.3) and hence we call $\tilde{\nabla}$ a semi-symmetric metric connection.

Let (M,g) and (B,g') be two Riemannian manifolds of dimension m and n with $m > n$. A Riemannian submersion is a smooth map $\pi : (M,g) \to (B,g')$ which is onto and satisfies the following conditions:

(i) π has maximal rank

(ii) The differential π_*preserves the lengths of horizontal vectors.

For each $q \in B$, $\pi^{-1}(q)$ is an $(m-n)-$dimensional submanifold of M. The submanifolds $\pi^{-1}(q)$ are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and π-related to a vector field X'on B. Note that we denote the projection morphisms on the distributions $\ker \pi_*$ and $(\ker \pi_*)^{\perp}$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

We recall that the sections of $\mathcal{V}$ and $\mathcal{H}$ are called the vertical vector fields and horizontal vector fields, respectively. A Riemannian submersion π: $M \to B$ determines two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on M, by the formulas:

$$\mathcal{T}(E,F) = \mathcal{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{2.4}$$

and

$$\mathcal{A}(E,F) = \mathcal{A}_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{2.5}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projections. These tensors are called O'Neill tensors.

From (2.4) and (2.5), one can obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W,$$

$$\nabla_V X = \mathcal{T}_V X + h(\nabla_V X),$$
$$\nabla_X V = v(\nabla_X V) + \mathcal{A}_X V,$$
$$\nabla_X Y = \mathcal{A}_X Y + h(\nabla_X Y)$$

for any $X, Y \in \Gamma(\ker \pi_*)^{\perp}$and $V, W \in \Gamma(\ker \pi_*)$. Moreover, if X is basic then

$$h(\nabla_v X) = \mathcal{A}_X V.$$

We note that for $U, V \in \Gamma(\ker \pi_*)$, $\mathcal{T}_U V$ coincides with the second fundamental form of the immersion of the fiber submanifolds. Moreover, for all $X, Y \in \Gamma(\ker \pi_*)^{\perp}$, $\mathcal{A}_X Y = \frac{1}{2} v\,[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that $\mathcal{A}$ is alternating on the horizontal distribution $\mathcal{A}_X Y = -\mathcal{A}_Y X$. T is symetric on the vertical distribution $T_U V = T_V U$ for $U, V \in \Gamma(\ker\pi_*)$

Lemma 2.1. If $\pi : (M, g) \to (B, g')$ is a Riemannian submersion X and $Y$are basic vector fields on M, π-related to X' and Y' on B, then we have the following properties:

(1)$h[X, Y]$ is a basic vector field and $\pi_* h\,[X, Y] = [X', Y'] \circ \pi$

(2) $h(\nabla_X Y)$ is a basic vector field π-related to $(\nabla'_{X'} Y')$where ∇ and ∇' are the Levi-Civita connection on M and B;

(3) $[E, U] \in \Gamma(\ker \pi_*)$, for any $U \in \Gamma(\ker \pi_*)$ and for any basic vector field E [7].

3 Riemannian submersions with a semi-symmetric metric connection

In this section, we introduce Riemannian submersions from a Riemannian manifold with a semi-symmetric metric connection onto a Riemannian manifold. We define and study O'Neill tensors with semi-symmetric metric connection. We show that this tensors are not skew symmetric.

Firstly, let (M, g) be a Riemannian manifold and $\tilde{\nabla}$ is a semi-symmetric metric connection on M. Let also π be a Riemannian submersion from M onto a Riemannian manifold B. Then the tensor $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{A}}$ with type $(1, 2)$ on M with respect to $\tilde{\nabla}$ is given by

$$\widetilde{\mathcal{T}}(E, F) = \widetilde{\mathcal{T}}_E F = h\tilde{\nabla}_{vE} vF + v\tilde{\nabla}_{vE} hF$$

for all $E, F \in \Gamma(TM)$.

By using (2.1) we obtain,

$$\widetilde{\mathcal{T}}_E F = \mathcal{T}_E F + \eta(hF) vE - g(vE, vF) hP. \tag{3.1}$$

Moreover the tensor $\tilde{A}$ with type (1,2) on M with respect to $\tilde{\nabla}$ is given by

$$\widetilde{\mathcal{A}}(E, F) = \widetilde{\mathcal{A}}_E F = v\tilde{\nabla}_{hE} hF + h\tilde{\nabla}_{hE} vF \;\; E, F \in \Gamma(TM).$$

for all $E, F \in \Gamma(TM)$.

In a similar way, using (2.1), we have

$$\widetilde{\mathcal{A}}_E F = \mathcal{A}_E F + \eta(vF) hE - g(hE, hF) vP. \tag{3.2}$$

In the sequel, we show that the tensor fields and $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{T}}$ are not skew symmetric.

Lemma 3.1. Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. Then we have

$$\begin{aligned} g(\widetilde{\mathcal{T}}_E F, G) &= g(\widetilde{\mathcal{T}}_E G, F) + 2g(\mathcal{T}_E F, G) + \eta(hF)g(vE, vG) - \eta(hG)g(vE, vF) \\ &\quad -g(vE, vF)g(hP, hG) + g(vE, vG)g(hP, hF) \end{aligned} \tag{3.3}$$

and

$$g(\widetilde{\mathcal{A}}_E F, G) = g(\widetilde{\mathcal{A}}_E G, F) + 2g(\mathcal{A}_E G, F) + 2\eta(vF)g(hE, hG) - 2\eta(vG)g(hE, hG) \tag{3.4}$$

for all $E, F, G \in \Gamma(TM)$.

Proof. For all $E, F, G \in \Gamma(TM)$ using (3.1) we have

$$g(\widetilde{\mathcal{T}}_E F, G) = g(\mathcal{T}_E F + \eta(hF)vE - h(g(vE, vF)P), G)$$

We know that $G = vG + hG$. Then we get,

$$g(\widetilde{\mathcal{T}}_E F, G) = g(\mathcal{T}_E F, G) + \eta(hF)g(vE, vG) - g(vE, vF)g(hP, hG) \tag{3.5}$$

In a similar way, we have

$$g(\widetilde{\mathcal{T}}_E G, F) = -g(\mathcal{T}_E G, F) + \eta(hG)g(vE, vF) - g(vE, vG)g(hP, hF) \tag{3.6}$$

Substracting (3.5) from (3.6), which gives (3.1). By using the same way, one can easily get (3.4). This completes the proof of the lemma. Q.E.D.

We now check symmetry properties of $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{A}}$.

Proposition 3.2. Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. Then we have

$$\widetilde{\mathcal{T}}_U W = \widetilde{\mathcal{T}}_W U \tag{3.7}$$

$$\widetilde{\mathcal{T}}_U X = \widetilde{\mathcal{T}}_X U + \eta(hX)vU \tag{3.8}$$

$$\widetilde{\mathcal{A}}_X Y = \widetilde{\mathcal{A}}_Y X + 2\mathcal{A}_X Y \tag{3.9}$$

$$\widetilde{\mathcal{A}}_X U = \widetilde{\mathcal{A}}_U X + 2\mathcal{A}_X U + \eta(vU)hX \tag{3.10}$$

for all $U, V \in \Gamma(\mathcal{V})$ $X, Y \in \Gamma(\mathcal{H})$

Proof. Since $\widetilde{\mathcal{T}}_U V = \mathcal{T}_U V$, for $U, V \in \Gamma(\mathcal{V})$ and $\mathcal{T}$ is symmetric on the vertical distribution, we get (3.7). In a similar way, since $\widetilde{\mathcal{A}}_X Y = \mathcal{A}_X Y$ for $X, Y \in \Gamma(\mathcal{H})$ and A is anti-symmetric on the horizontal distribution, we obtain (3.9).

For all $U \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$ using (3.1) we have

$$\widetilde{\mathcal{T}}_U X = \mathcal{T}_U X + \eta(hX)vU - g(vU, vX)hP$$

and

$$\widetilde{\mathcal{T}}_X U = \mathcal{T}_X U + \eta(hU)vX - g(vX, vU)hP.$$

Then, using (3.1), (2.1), $vX = 0$ and $T_U X = T_X U$, we have

$$\widetilde{\mathcal{T}}_U X = \widetilde{\mathcal{T}}_X U + \eta(hX)vU.$$

On the other hand, using (3.2), (2.1), $vX = 0$, $hU = 0$ and $\mathcal{A}_U X = -\mathcal{A}_X U$ we get

$$\widetilde{\mathcal{A}}_X U = \widetilde{\mathcal{A}}_U X + 2\mathcal{A}_X U + \eta(vU)hX$$

which completes proof. Q.E.D.

Lemma 3.3. Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. Then we have

$$\tilde{\nabla}_V W = \hat{\nabla}_V W + \mathcal{T}_V W \tag{3.11}$$

$$\tilde{\nabla}_V X = h\nabla_V X + \mathcal{T}_V X + \eta(X)V \tag{3.12}$$

$$\tilde{\nabla}_X V = \mathcal{A}_X V + v\tilde{\nabla}_X V + \eta(V)X \tag{3.13}$$

$$\tilde{\nabla}_X Y = \mathcal{A}_X Y + h\tilde{\nabla}_X Y + g(X, Y)vP \tag{3.14}$$

for all $V, W \in \Gamma(\mathcal{V})$ and $X, Y \in \Gamma(\mathcal{H})$, where $\hat{\nabla}_V W = v\tilde{\nabla}_V W$ and X is basic then, $h\tilde{\nabla}_V X = h\tilde{\nabla}_X V = \mathcal{A}_X V$.

Proof. Since ∇ is a Levi-Civita connection, using (2.1), we obtain (3.11). The other assertions can be obtain in a similar way. Q.E.D.

Definition 3.4. Let$\pi : (M, g) \to (B, g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. The covariant derivates of $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{T}}$ given by

$$(\widetilde{\nabla}_E \widetilde{\mathcal{A}})_F H = (\widetilde{\nabla}_E \widetilde{\mathcal{A}})(F, H) = \widetilde{\nabla}_E(\widetilde{\mathcal{A}}_F H) - \widetilde{\mathcal{A}}_{\widetilde{\nabla}_E F}(H) - \widetilde{\mathcal{A}}_F(\widetilde{\nabla}_E H) \tag{3.15}$$

and

$$(\widetilde{\nabla}_E \widetilde{\mathcal{T}})_F H = (\widetilde{\nabla}_E \widetilde{\mathcal{T}})(F, H) = \widetilde{\nabla}_E(\widetilde{\mathcal{T}}_F H) - \widetilde{\mathcal{T}}_{\widetilde{\nabla}_E F}(H) - \widetilde{\mathcal{T}}_F(\widetilde{\nabla}_E H) \tag{3.16}$$

for all $E, F, H \in \Gamma(TM)$.

Theorem 3.5. Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. The covariant derivates of $\widetilde{A}$ and $\widetilde{T}$ given by

$$(\widetilde{\nabla}_V \widetilde{\mathcal{A}})_W E = -\widetilde{\mathcal{A}}_{T_V W} E$$

$$(\widetilde{\nabla}_V \widetilde{\mathcal{T}})_Y E = -\widetilde{\mathcal{T}}_{T_V Y + \eta(Y)V} E$$

$$(\widetilde{\nabla}_X \widetilde{\mathcal{A}})_W E = -\widetilde{\mathcal{A}}_{\mathcal{A}_X W + \eta(W)X} E$$

$$(\widetilde{\nabla}_X \widetilde{\mathcal{T}})_Y E = -\widetilde{\mathcal{T}}_{\mathcal{A}_X Y + g(X,Y)vP} E$$

for all $X, Y \in \Gamma(\mathcal{H})$ and $V, W, E \in \Gamma(\mathcal{V})$

Proof. Firstly, since A is a horizontal vector field, the vertical component is ignored. Therefor from (3.15) we get

$$(\widetilde{\nabla}_V\widetilde{\mathcal{A}})_W E = -\widetilde{\mathcal{A}}_{\widetilde{\nabla}_V W}E.$$

Then using (3.11), we have

$$(\widetilde{\nabla}_V\widetilde{\mathcal{A}})_W E = -\widetilde{\mathcal{A}}_{T_V W}E.$$

Similarly, since T is vertical vector field, the horizontal components are ignored. Therefore, from (3.13), we arrive

$$(\widetilde{\nabla}_V\widetilde{\mathcal{T}})_Y E = -\widetilde{\mathcal{T}}_{\widetilde{\nabla}_V Y}E.$$

Therefore using (3.12) we conclude

$$(\widetilde{\nabla}_V\widetilde{\mathcal{T}})_Y E = -\widetilde{\mathcal{T}}_{\mathcal{T}_V Y+\eta(Y)V}E.$$

On the other hand, considering that $\widetilde{\mathcal{A}}$ is a horizontal vector, using (3.13), we get

$$(\widetilde{\nabla}_X\widetilde{\mathcal{A}})_w E = -\widetilde{\mathcal{A}}_{\widetilde{\nabla}_X W}(E).$$

Then from (3.13), we have

$$(\widetilde{\nabla}_X\widetilde{\mathcal{A}})_W E = -\widetilde{\mathcal{A}}_{\mathcal{A}_X W+\eta(W)X}E.$$

Finally, using (3.16), we conclude

$$(\widetilde{\nabla}_X\widetilde{\mathcal{T}})_Y E = -\widetilde{\mathcal{T}}_{\widetilde{\nabla}_X Y}(E).$$

Therefore from (3.14), we obtain

$$(\widetilde{\nabla}_X\widetilde{\mathcal{T}})_Y E = -\widetilde{\mathcal{T}}_{\mathcal{A}_X Y+g(X,Y)vP}E.$$

Q.E.D.

4 Curvature relations with respect to semi-symmetric metric connection

In this section, we will show for the first time how to obtain curvature tensors defined in the Riemannian manifold in semi-symmetric metric connection. We first note that we denote curvature tensor fields of $\widetilde{\nabla}$ by $\widetilde{R}$

$$\widetilde{R}(U,V)W = \widetilde{\nabla}_U\widetilde{\nabla}_V W - \widetilde{\nabla}_V\widetilde{\nabla}_U W - \widetilde{\nabla}_{[U,V]}W. \tag{4.1}$$

Theorem 4.1. Let $\pi : (M,g) \to (B,g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. We denote Riemannian curvatures of M and any fibre by $\widetilde{R}$ and $\hat{R}$, respectively. Then we have

$$g(\widetilde{R}(U,V)W,F) = g(\hat{R}(U,V)W,F) + g(\mathcal{T}_V W,\mathcal{T}_U F) - g(\mathcal{T}_U W,\mathcal{T}_V F) + \eta(\mathcal{T}_U W)g(U,F) - \eta(\mathcal{T}_U W)g(V,F) \tag{4.2}$$

and

$$g(\widetilde{R}(U,V)W,X) = g((\widetilde{\nabla}_U\mathcal{T}_V)W,X) - g((\widetilde{\nabla}_V\mathcal{T}_U)W,X) \tag{4.3}$$

for all $U,V,W,F \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$.

Proof. For all $U, V, W, F \in \Gamma(\mathcal{V})$ using (3.14) and (3.12), we arrive

$$\widetilde{\nabla}_U \widetilde{\nabla}_V W = \mathcal{T}_U \mathcal{T}_V W + h\widetilde{\nabla}_U \mathcal{T}_V W + \eta(\mathcal{T}_V W)U + \mathcal{T}_U \hat{\nabla}_V W + \widetilde{\nabla}_U \hat{\nabla}_V W.$$

and

$$\widetilde{\nabla}_{[U,V]} W = \mathcal{T}_{[U,V]} W + \hat{\nabla}_{[U,V]} W.$$

Then, from (4.1) we have

$$\begin{aligned}\widetilde{R}(U,V)W =& \mathcal{T}_U \mathcal{T}_V W + h\widetilde{\nabla}_U \mathcal{T}_V W + \eta(\mathcal{T}_V W)U + \mathcal{T}_U \hat{\nabla}_V W + \widetilde{\nabla}_U \hat{\nabla}_V W - \mathcal{T}_V \mathcal{T}_U W - h\widetilde{\nabla}_V \mathcal{T}_U W \\ & - \eta(\widetilde{\nabla}_U W)V - \mathcal{T}_V \hat{\nabla}_U W - \widetilde{\nabla}_V \hat{\nabla}_U W - \mathcal{T}_{[U,V]} W - \hat{\nabla}_{[U,V]} W.\end{aligned}$$

If this last equation is multiplied by $V \in \Gamma(\mathcal{V})$ we have

$$g(\widetilde{R}(U,V)W,F) = g(\hat{R}(U,V)W,F) + g(\mathcal{T}_V W, \mathcal{T}_U F) - g(\mathcal{T}_U W, \mathcal{T}_V F) + \eta(\mathcal{T}_U W)g(U,F) - \eta(\mathcal{T}_U W)g(V,F)$$

Similarly, for all $X \in \mathcal{H}$, we obtain

$$g(\widetilde{R}(U,V)W,X) = g((\widetilde{\nabla}_U \mathcal{T}_V)W,X) - g((\widetilde{\nabla}_V \mathcal{T}_U)W,X)$$

Q.E.D.

Corollary 4.2. Let $\pi : (M,g) \to (B,g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. We denote Riemannian curvatures of M and any fibre by $\widetilde{R}$ and $\hat{R}$, respectively. Then we have

$$\begin{aligned}\widetilde{R}(V,W,X,K) \quad = \quad & \hat{R}(V,W,X,K) + g(\mathcal{T}_V h\widetilde{\nabla}_W X,K) - g(\mathcal{T}_W h\widetilde{\nabla}_V X,K) \\ & +\eta(h\widetilde{\nabla}_W X)g(V,K) - \eta(h\widetilde{\nabla}_V X)g(W,K) + g(\hat{\nabla}_V \mathcal{T}_W X,K) \\ & -g(\hat{\nabla}_W \mathcal{T}_V X,K) + g(h\widetilde{\nabla}_V X,hP)g(W,K) - g(h\widetilde{\nabla}_W X,hP)g(V,K) \\ & +g(\mathcal{T}_V X,vP)g(W,K) - g(\mathcal{T}_W X,vP)g(V,K) \\ & +\eta(X)g(V,vP)g(W,K) - \eta(X)g(W,vP)g(V,K) \\ & -g(\mathcal{T}_{[V,W]}X,K) - \eta(X)g([V,W],K) + g(X,\nabla_V P)g(W,K) \\ & -g(X,\nabla_W P)g(V,K) + \eta(X)\eta(V)g(W,K) - \eta(X)\eta(W)g(V,K)\end{aligned}$$

and

$$\begin{aligned}\widetilde{R}(X,V,Y,Z) \quad = \quad & \hat{R}(X,V,Y,W) + g(\mathcal{A}_X \mathcal{T}_V Y,Z) + \eta(\mathcal{T}_V Y)g(X,Z) \\ & +\eta(X)\eta(V)g(X,Z) - g(\mathcal{T}_V \mathcal{A}_X Y,Z) - g(X,Y)g(\mathcal{T}_V vP,Z) \\ & -\eta(Y)g([X,V],Z) + \eta(Y)g(\mathcal{A}_V X,Z)\end{aligned}$$

for all $V, W, K \in \Gamma(\mathcal{V})$ and $X, Y, Z \in \Gamma(\mathcal{H})$.

Theorem 4.3. Let $\pi : (M,g) \to (B,g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. We denote Riemannian curvatures of M and B by $\widetilde{R}$ and R', respectively. Then we have

$$\begin{aligned}g(\widetilde{R}(X,Y)Z,H) \quad = \quad & g(R'(X,Y)Z,H) - g(\mathcal{A}_Y Z, \mathcal{A}_X H) + \eta(\mathcal{A}_Y Z)g(X,H) + g(\mathcal{A}_X Z, \mathcal{A}_Y H) \\ & -\eta(\mathcal{A}_X Z)g(Y,H) + g(Y,Z)g(\widetilde{\nabla}_X P,H) - g(X,Z)g(\widetilde{\nabla}_Y P,H) \qquad (4.4)\end{aligned}$$

and

$$g(\widetilde{R}(X,Y)Z,V) = g(R'(X,Y)Z,V) + g(\mathcal{A}_Y Z,\mathcal{A}_X V) + g(v\widetilde{\nabla}_X \mathcal{A}_Y Z,V) + g(\mathcal{A}_X h\widetilde{\nabla}_Y Z,V) - g(\mathcal{A}_X Z,\mathcal{A}_Y V) - g(v\widetilde{\nabla}_Y \mathcal{A}_X Z,V) - g(\mathcal{A}_Y h\widetilde{\nabla}_X Z,V) \quad (4.5)$$

for all $X,Y,Z,H \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$.

Proof. For all $X,Y,Z \in \Gamma(\mathcal{H})$,using (3.13), (3.14) and (4.1) we have

$$\widetilde{R}(X,Y)Z = R'(X,Y)Z + \mathcal{A}_X\mathcal{A}_Y Z + v\widetilde{\nabla}_X \mathcal{A}_Y Z + \eta(\mathcal{A}_Y Z)X + \mathcal{A}_X h\widetilde{\nabla}_Y Z - \mathcal{A}_Y\mathcal{A}_X Z - v\widetilde{\nabla}_Y \mathcal{A}_X Z - \eta(\mathcal{A}_X Z)Y - \mathcal{A}_Y h\widetilde{\nabla}_X Z - \mathcal{A}_{[X,Y]}Z$$

Then for all $H \in \Gamma(\mathcal{H})$ we conclude

$$g(\widetilde{R}(X,Y)Z,H) = g(R'(X,Y)Z,H) - g(\mathcal{A}_Y Z,\mathcal{A}_X H) + \eta(\mathcal{A}_Y Z)g(X,H) + g(\mathcal{A}_X Z,\mathcal{A}_Y H) - \eta(A_X Z)g(Y,H) + g(Y,Z)g(\widetilde{\nabla}_X P,H) - g(X,Z)g(\widetilde{\nabla}_Y P,H)$$

On the other hand, for all $V \in \Gamma(\mathcal{V})$ we obtain

$$g(\widetilde{R}(X,Y)Z,V) = g(R'(X,Y)Z,V) + g(\mathcal{A}_Y Z,\mathcal{A}_X V) + g(v\widetilde{\nabla}_X \mathcal{A}_Y Z,V) + g(\mathcal{A}_X h\widetilde{\nabla}_Y Z,V) - g(\mathcal{A}_X Z,\mathcal{A}_Y V) - g(v\widetilde{\nabla}_Y \mathcal{A}_X Z,V) - g(\mathcal{A}_Y h\widetilde{\nabla}_X Z,V)$$

Q.E.D.

Corollary 4.4. Let $\pi : (M,g) \to (B,g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. We denote Riemannian curvatures of M and any fibre by $\widetilde{R}$ and $\hat{R}$. Then we have

$$\widetilde{R}(V,W,X,Y) = \widehat{R}(V,W,X,Y) + g(\mathcal{T}_W X,\mathcal{T}_V Y) - g(\mathcal{T}_V X,\mathcal{T}_W Y) \eta(X)g(\mathcal{T}_V W,Y) - \eta(X)g(\mathcal{T}_W V,Y),$$

$$\begin{aligned}\widetilde{R}(V,X,W,Y) = {} & \widehat{R}(V,X,W,Y) + g(h\widetilde{\nabla}_V \mathcal{A}_X W,Y) + g(\mathcal{T}_V v\widetilde{\nabla}_X W,Y) \\ & + g(\hat{\nabla}_V W,P)g(X,Y) + g(\mathcal{T}_V W,P)g(X,Y) \\ & + g(W,\nabla_V P)g(X,Y) + g(W,V)g(X,Y) \\ & - \eta(V)\eta(W)g(X,Y) + \eta(W)g(h\widetilde{\nabla}_V X,Y) \\ & + \eta(W)g(\mathcal{T}_V X,Y) - g(\mathcal{A}_X \hat{\nabla}_V W,Y) - \eta(\hat{\nabla}_V W)g(X,Y) \\ & - g(h\widetilde{\nabla}_X \mathcal{T}_V W,Y) - g(\mathcal{T}_{[V,X]}W,Y),\end{aligned}$$

$$\begin{aligned}\widetilde{R}(X,V,Y,W) = {} & \widehat{R}(X,V,Y,W) + g(\mathcal{A}_X h\widetilde{\nabla}_V Y,W) + g(X,h\widetilde{\nabla}_V Y)g(vP,W) \\ & + g(v\widetilde{\nabla}_X T_V Y,W) + g(\mathcal{A}_X Y,vP)g(V,W) \\ & + g(h\widetilde{\nabla}_X Y,hP)g(V,W) + g(X,Y)g(V,W) \\ & + g(Y,\widetilde{\nabla}_X P)g(V,W) + g(Y,X)g(V,W) \\ & - \eta(X)\eta(Y)g(V,W) + \eta(Y)g(v\widetilde{\nabla}_X V,W) - g(\hat{\nabla}_V \mathcal{A}_X Y,W) \\ & - g(\mathcal{T}_V h\widetilde{\nabla}_X Y,W) - \eta(h\widetilde{\nabla}_X Y)g(V,W) - g(h\widetilde{\nabla}_V X,Y)g(vP,W) \\ & - g(X,h\widetilde{\nabla}_V Y)g(vP,W) - g(X,Y)g(\hat{\nabla}_V vP,W) - g(\mathcal{T}_{[X,V]}Y,W)\end{aligned}$$

and

$$\widetilde{R}(X,Y,V,W) = \widehat{R}(X,Y,V,W) + g(\mathcal{A}_X\mathcal{A}_YV,W) - g(\mathcal{A}_Y\mathcal{A}_XV,W) + g(X,\mathcal{A}_YV)\eta(W) - g(Y,\mathcal{A}_XV)\eta(W) + \eta(V)g(\mathcal{A}_XY,W) - \eta(V)g(\mathcal{A}_YX,W)$$

for all $V,W \in \Gamma(\mathcal{V})$ and $X,Y \in \Gamma(\mathcal{H})$

Theorem 4.5. Let $\pi : (M,g) \to (B,g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. We denote Riemannian curvatures of M, B and any fibre by $\widetilde{R}, R'$ and $\hat{R}$, respecticely. Then we have

$$\widetilde{K}(U,V) = g(\hat{R}(U,V)U,V) - \|\mathcal{T}_UV\|^2 + g(\mathcal{T}_VV,\mathcal{T}_UU) + \eta(\mathcal{T}_VU)g(U,V) - \eta(\mathcal{T}_UU)$$

and

$$\widetilde{K}(X,Y) = K'(X,Y) - g(\mathcal{A}_YY,\mathcal{A}_XX) - \|\mathcal{A}_XY\|^2 + \eta(\mathcal{A}_YY) + g(\widetilde{\nabla}_XP,X) + g(X,Y)(-\eta(\mathcal{A}_XY) - g(\widetilde{\nabla}_YP,X))$$

for all $U,V \in \Gamma(\mathcal{V})$ and $X,Y \in \Gamma(\mathcal{H})$.

Proof. For all $U,V \in \Gamma(\mathcal{V})$ and $X,Y \in \Gamma(\mathcal{H})$ using (4.4) we conclude

$$\widetilde{K}(U,V) = g(\hat{R}(U,V)U,V) - g(\mathcal{T}_UV,\mathcal{T}_VU) + g(\mathcal{T}_VV,\mathcal{T}_UU) + \eta(\mathcal{T}_VU)g(U,V) - \eta(\mathcal{T}_UU) - g(V,V).$$

On the other hand, for all $X,Y \in \Gamma(\mathcal{H})$ we arrive

$$\widetilde{K}(X,Y) = g(R'(X,Y)X,Y) - g(\mathcal{A}_YY,\mathcal{A}_XX) + g(\mathcal{A}_XY,\mathcal{A}_YX) + \eta(\mathcal{A}_YY)g(X,X) - \eta(\mathcal{A}_XY)g(Y,X) + g(Y,Y)g(\widetilde{\nabla}_XP,X) - g(X,Y)g(\widetilde{\nabla}_YP,X)$$

which completes the proof. Q.E.D.

Corollary 4.6. Let $\pi : (M,g) \to (B,g')$ be a Riemannian submersion from a Riemannian manifold with semi-symmetric metric connection onto a Riemannian manifold. We denote Riemannian curvatures of M, B and any fibre by $\widetilde{R}, R'$ and $\hat{R}$, respectiely. Then we have

$$\widetilde{K}(V,X) = \hat{R}(V,X,V,X) + g(h\widetilde{\nabla}_V\mathcal{A}_XV,X) + g(\mathcal{T}_Vv\widetilde{\nabla}_XV,X) + \eta(\hat{\nabla}_VV) + \eta(\mathcal{T}_VV) + \eta(V)g(h\widetilde{\nabla}_VX,X) - g(\mathcal{A}_X\hat{\nabla}_VV,X) - \eta(\hat{\nabla}_VV) - g(h\widetilde{\nabla}_X\mathcal{T}_VV,X) - g(\mathcal{T}_{[V,X]}V,X) + g(V,\nabla_VP) + 1 - [\eta(V)]^2$$

and

$$\widetilde{K}(X,V) = g(\widehat{R}(X,V)X,V) + g(\mathcal{A}_Xh\widetilde{\nabla}_VX,V) + g(X,h\widetilde{\nabla}_VX)\eta(W) + g(v\widetilde{\nabla}_X\mathcal{T}_VX,V) + g(\mathcal{A}_XX,vP) + g(h\widetilde{\nabla}_XX,hP) + g(X,\nabla_XP) + 2 - [\eta(X)]^2 + \eta(X)g(v\widetilde{\nabla}_XV,V) - g(\hat{\nabla}_V\mathcal{A}_XX,V) - g(\mathcal{T}_Vh\widetilde{\nabla}_XX,V) - \eta(h\widetilde{\nabla}_XY) - g(h\widetilde{\nabla}_VX,X)\eta(V) - g(X,h\widetilde{\nabla}_VX)\eta(V) - g(\hat{\nabla}_VvP,V) - g(T_{[X,V]}X,V)$$

for all $V \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$.

Conclusion 4.7. Semi-symmetric metric connections and Riemannian submersions have potential for applications in many fields of mathematics and physics. Researchers have increased studies on this field from different areas in recent years. In this paper, the idea of examining Riemann submersion with different connections is emphasized. We defined and studied Riemannian submersions with semi-symmetric metric connection for the first time. The works on this subject will be useful tools for the applications of Riemannian submersion with different connections.

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